Longitudinal Phase Space Distortion in FFAGs

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Muon Collaboration Friday Meeting
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FFAG Longitudinal Equations of Motion

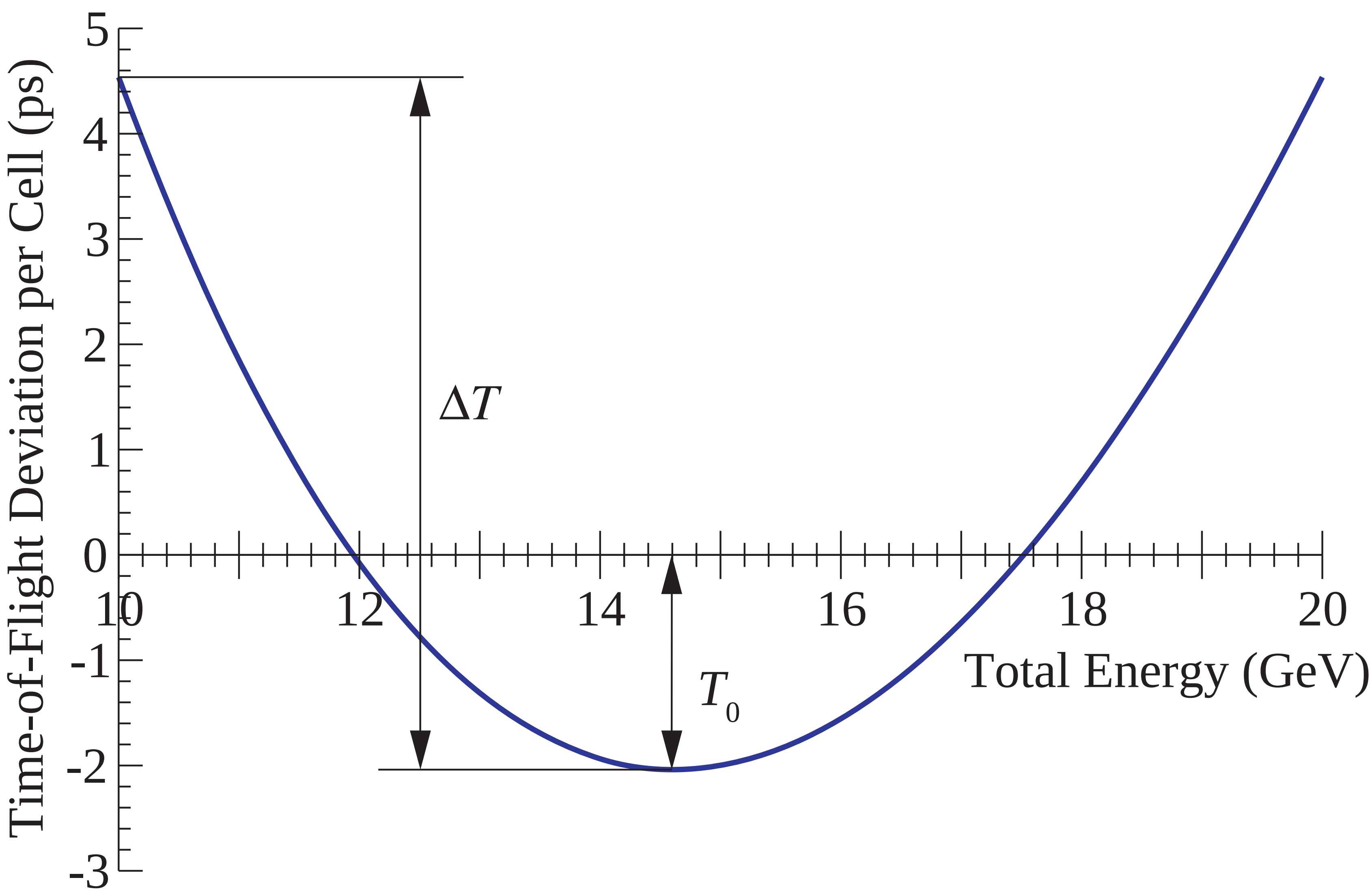
- Time of flight is approximately a parabolic function of energy

$$\frac{d\tau}{ds} = \Delta T \left(\frac{2E - E_i - E_f}{\Delta E}\right)^2 - T_0,$$

- Energy gain from RF

$$\frac{dE}{ds} = V\cos(\omega\tau),$$

Time-of-Flight vs. Energy



Normalized Variables

- Change of variables

$$x = \omega\tau \qquad p = \frac{E - E_i}{\Delta E} \qquad u = \frac{s}{\omega \Delta T}$$

  - Accelerate from $p = 0$ to $p = 1$

- New equations of motion

$$\frac{dx}{du} = (2p-1)^2 - b \qquad \frac{dp}{du} = a \cos x \qquad a = \frac{V}{\omega \Delta T \Delta E} \qquad b = \frac{T_0}{\Delta T}$$

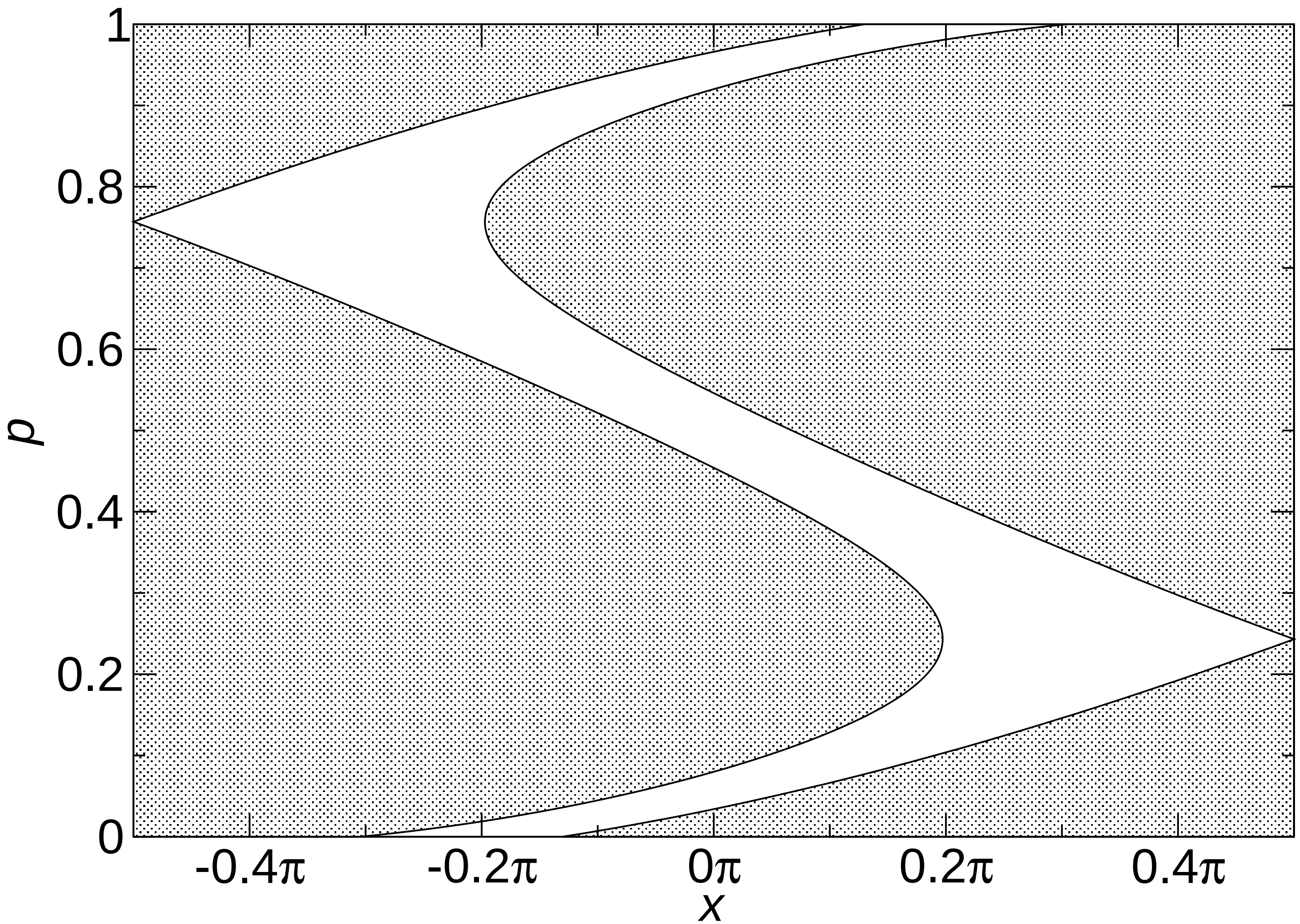
- Hamiltonian

$$\frac{1}{6}(2p-1)^3 - \frac{b}{2}(2p-1) - a \sin x$$

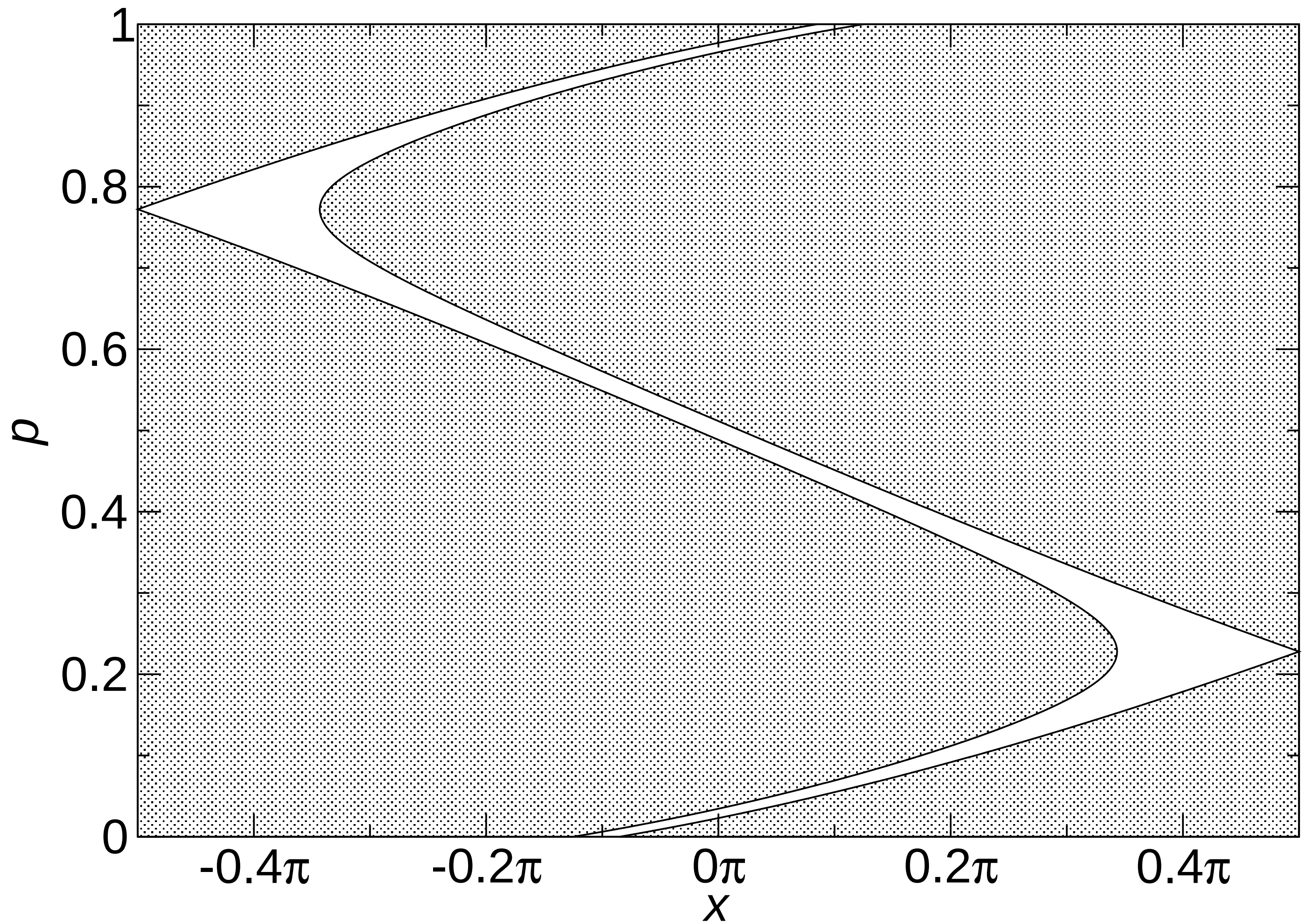
Parameter Regimes

- To pass particles through from $p = 0$ to $p = 1$, require $a > b^{3/2}/3$
- For central particle to cross $p = 0$ and $p = 1$, require $a > |1/6 - b/2|$
- Small a, smaller phase space region for bunch
- Requirements together lead to minimum a of $1/24$
  - Smaller a gives more emittance growth
- Based on design requirements (emittance, allowed emittance growth, etc.), determine a and b

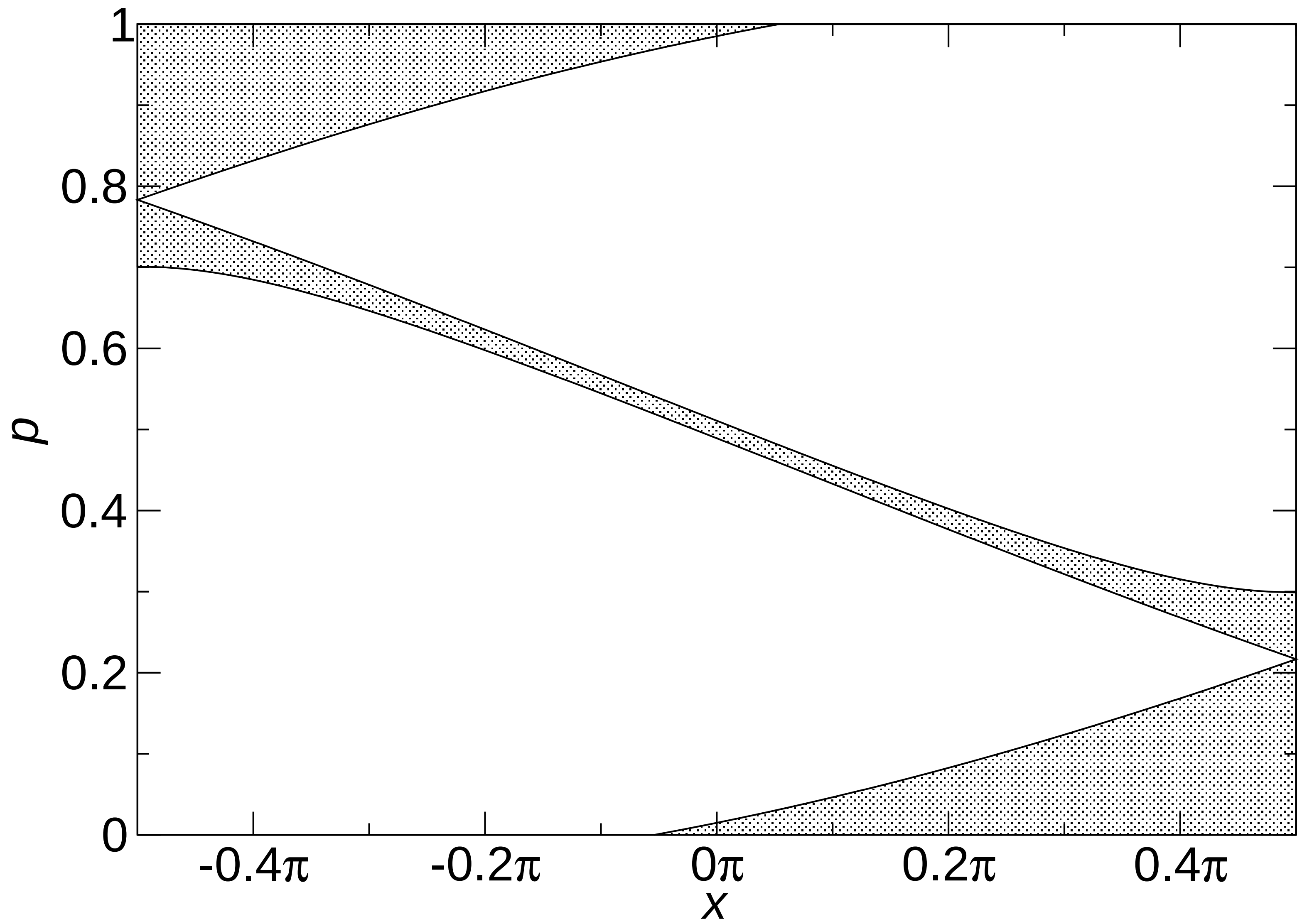
Particles Passing Through



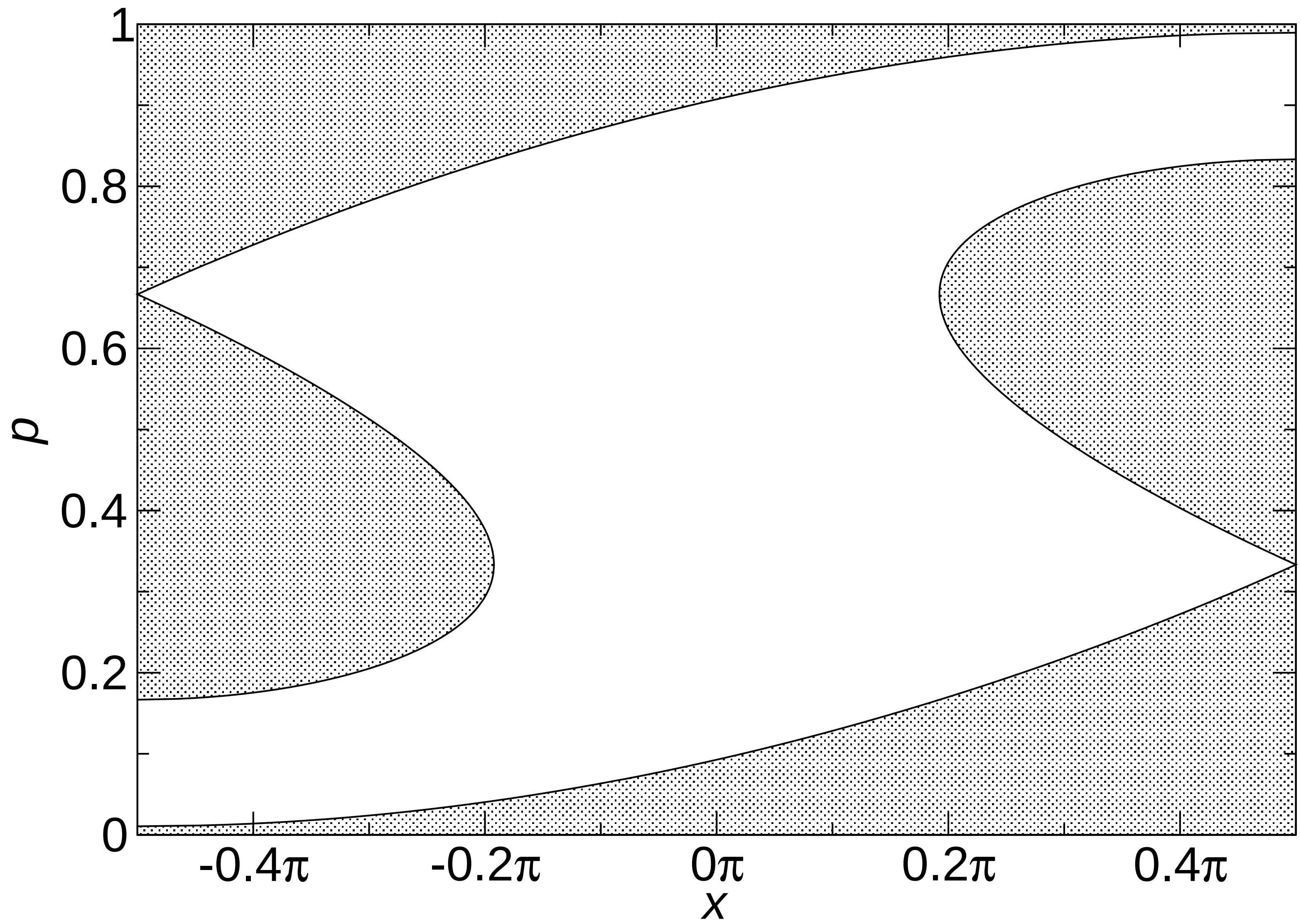
Particles Barely Pass



Particles Can't Pass



Central Particle Doesn't Make It



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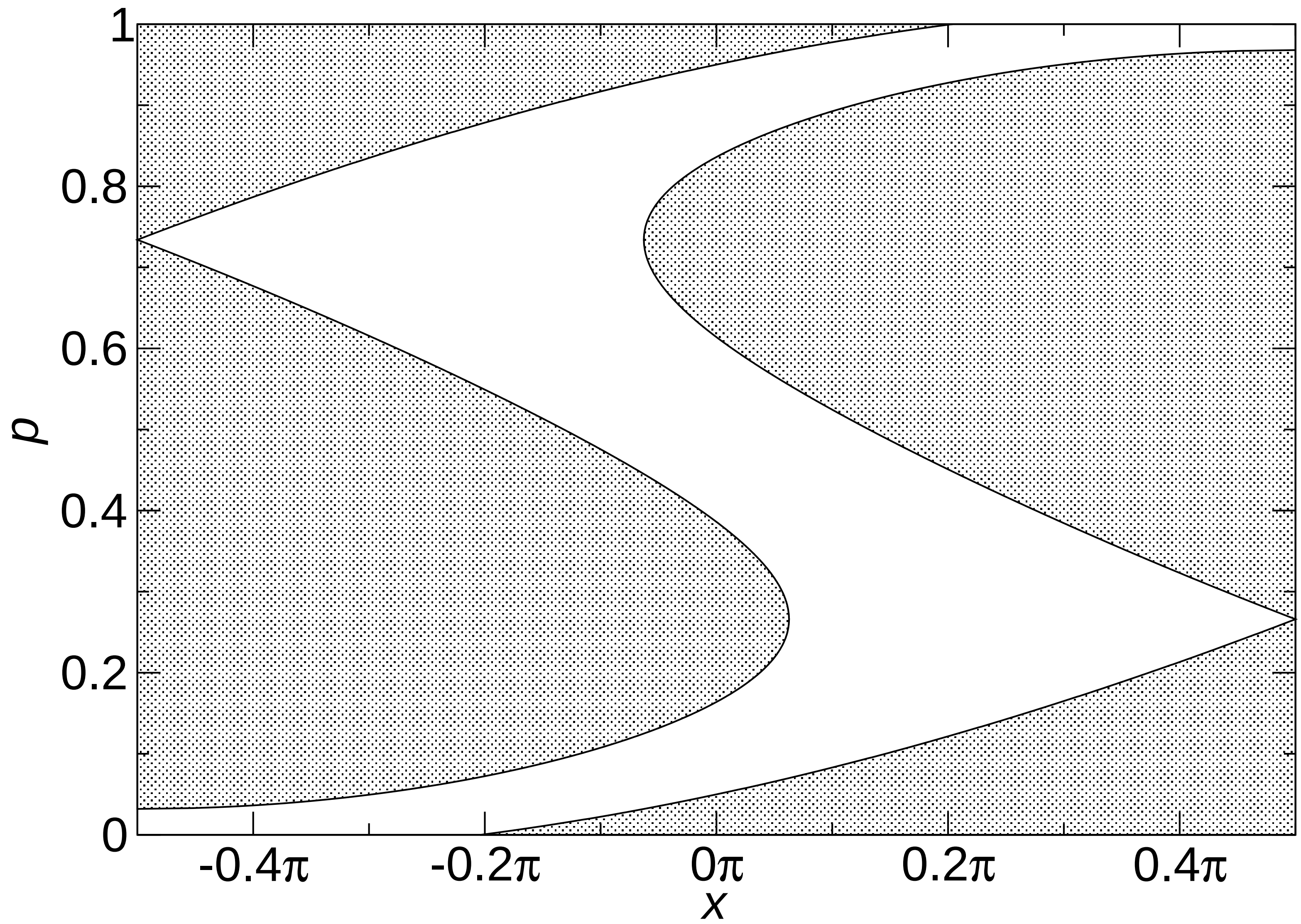
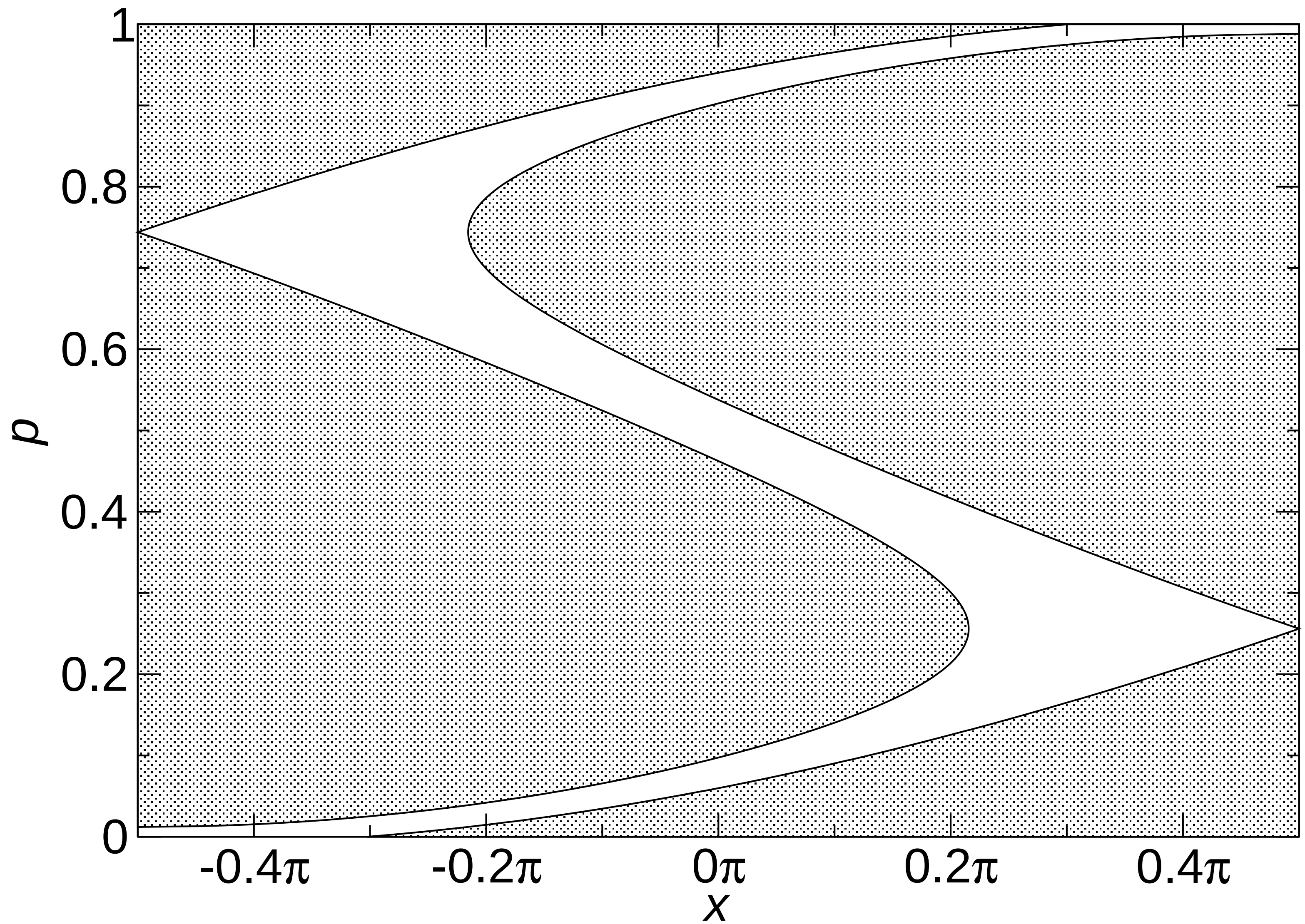
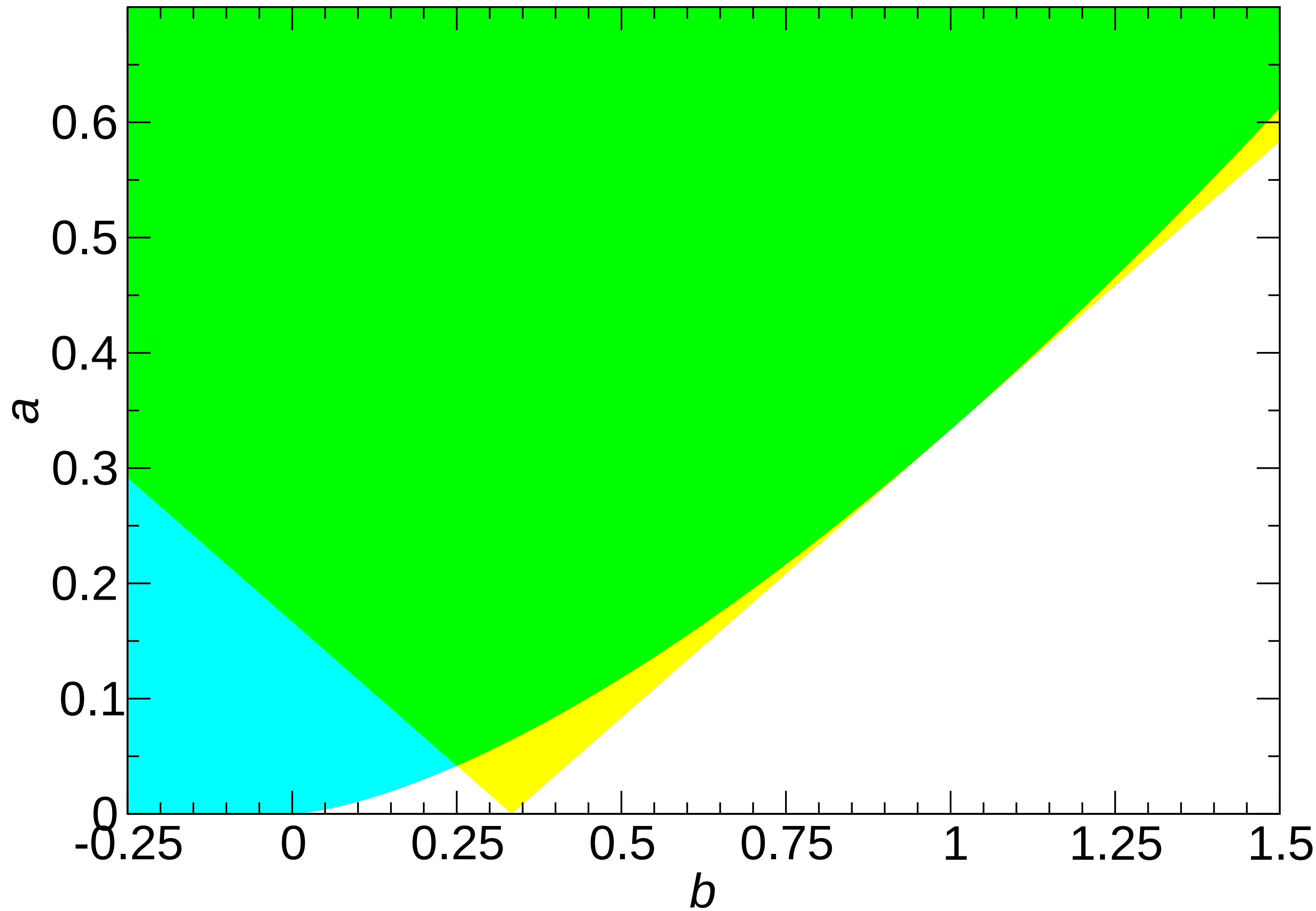


Lower a



Allowed Region of Parameter Space



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Symplectic Maps

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- A general symplectic map can be described by a “Dragt-Finn Factorization”:

$$e^{-:g_1:}\ldots e^{:f_4:}e^{:f_3:}e^{:f_2:}e^{:f_1:}$$

  - I won’t go into what precisely this means…
- f_n is a nth-order homogeneous polynomial in the phase space variables
- f_1 describes the final reference point, g_1 the initial reference point
- f_2 is the linear part of the map
- The rest are nonlinear

One-Dimensional Example

- Write f_n as

$$f_n = \sum_{k=0}^{n} f_{nk} x^{n-k} p^k$$

- Calculate the emittance using the second-order covariance matrix

$$\sqrt{\det\{\langle \boldsymbol{z}\boldsymbol{z}^T \rangle - \langle \boldsymbol{z} \rangle \langle \boldsymbol{z} \rangle^T\}}$$

- To lowest order, the emittance growth is ($f_2 = 0$)

$$\frac{3}{4}\langle J^2 \rangle (9f_{30}^2 - 6f_{30}f_{32} + 5f_{32}^2 + 9f_{33}^2 - 6f_{33}f_{31} + 5f_{31}^2) \\ - \frac{1}{2}\langle J \rangle^2 [(3f_{30} + f_{32})^2 + (3f_{33} + f_{31})^2]$$

  - $\langle J \rangle = \epsilon$ is the emittance; $\langle J^2 \rangle > \langle J \rangle^2$
  - This can be negative if $\langle J^2 \rangle < (4/3)\langle J \rangle^2$ (equality for uniform)!

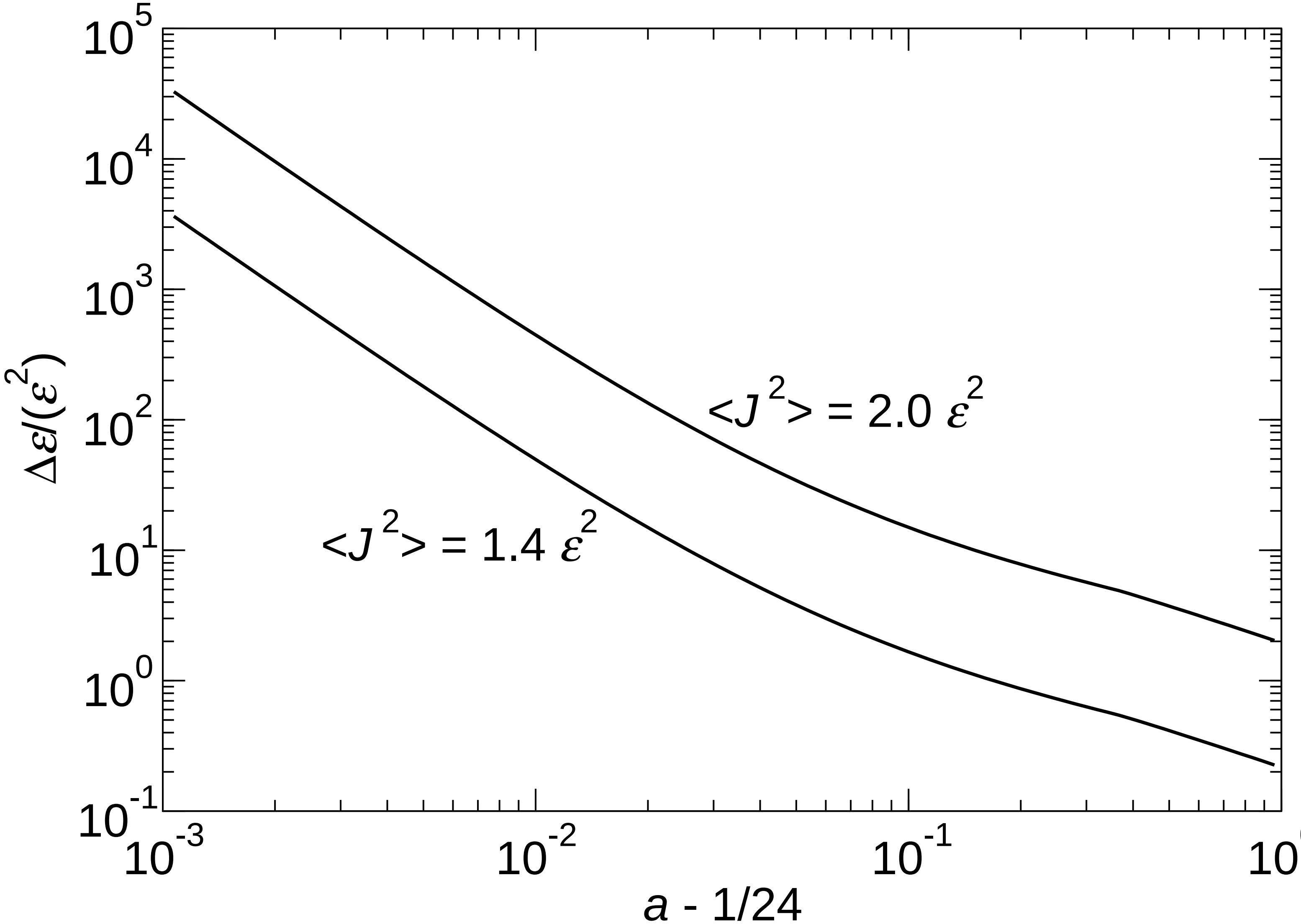
Computing Emittance Growth

- For given a and b, compute f_3
- Transform f_3 with a linear transform corresponding to the orientation of the incoming ellipse
  - Minimize emittance growth over that transform (two free parameters)
- Minimize the result with respect to b
- Have emittance growth as a function of a

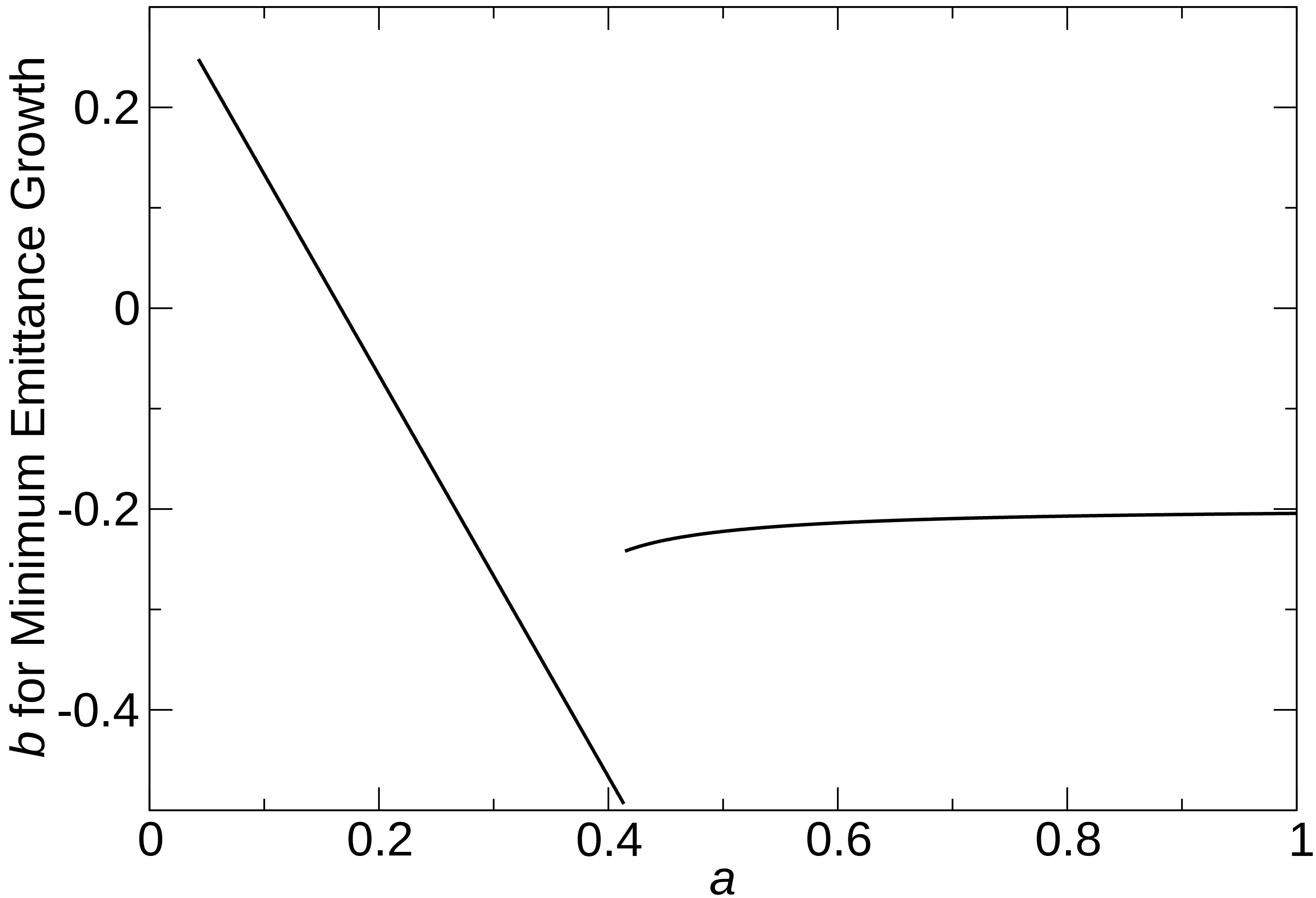
Emittance Growth Analysis

- For small a, $\Delta\epsilon/(\epsilon^2) \propto (a - 1/24)^{-2}$
- Emittance growth is smaller for smaller $\langle J^2 \rangle / \epsilon^2$
- To use:
  - Compute emittance in normalized coordinates
  - Choose acceptable emittance growth
  - Find a which gives that emittance growth
- Optimal b is independent of $\langle J^2 \rangle / \epsilon^2$
- For small a, optimal b is the minimum b
  - Can be negative!
- Optimal ellipse orientation is tilted, even though initial phase space trajectories are flat

Emittance Growth vs. a



Optimal b



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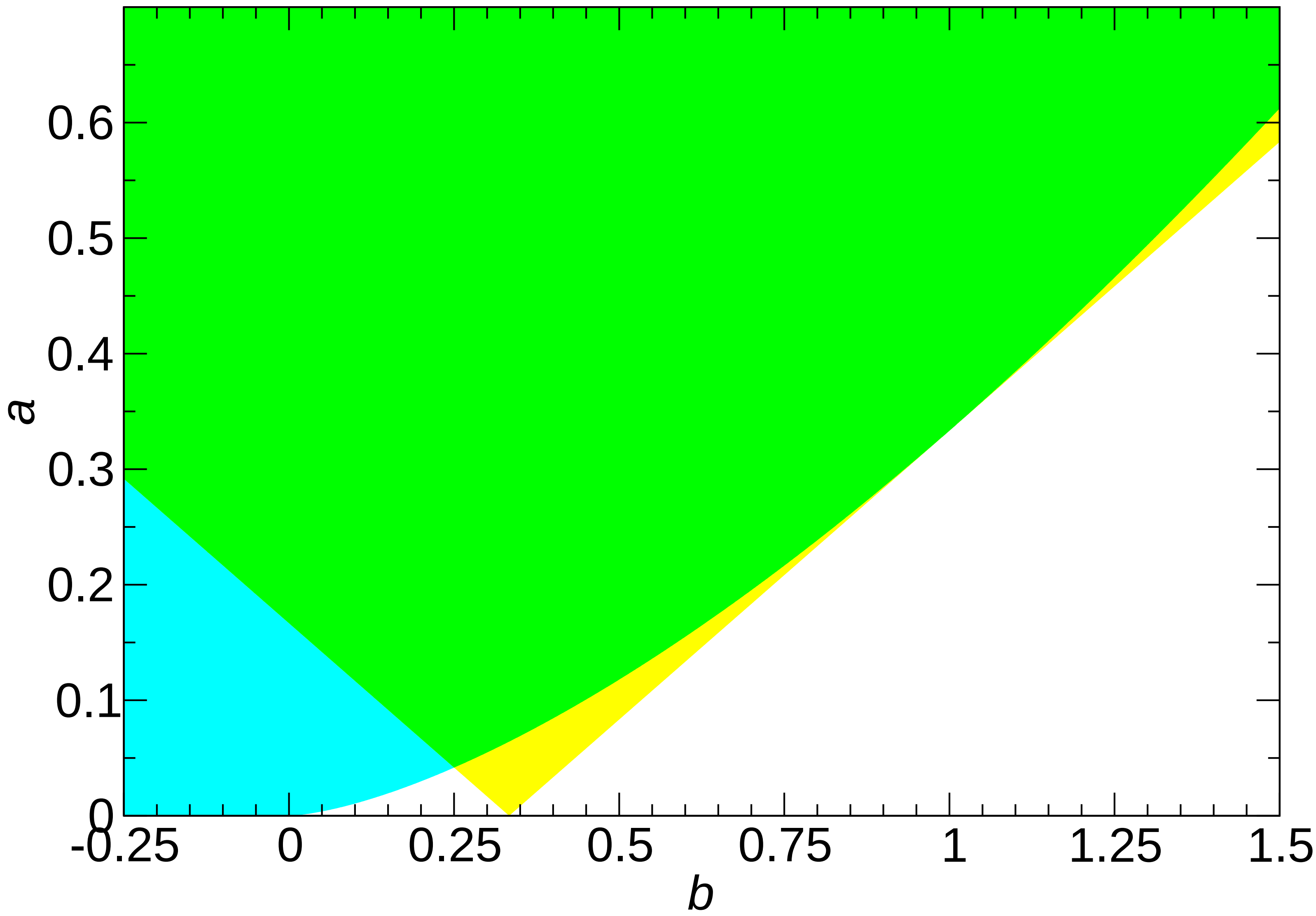
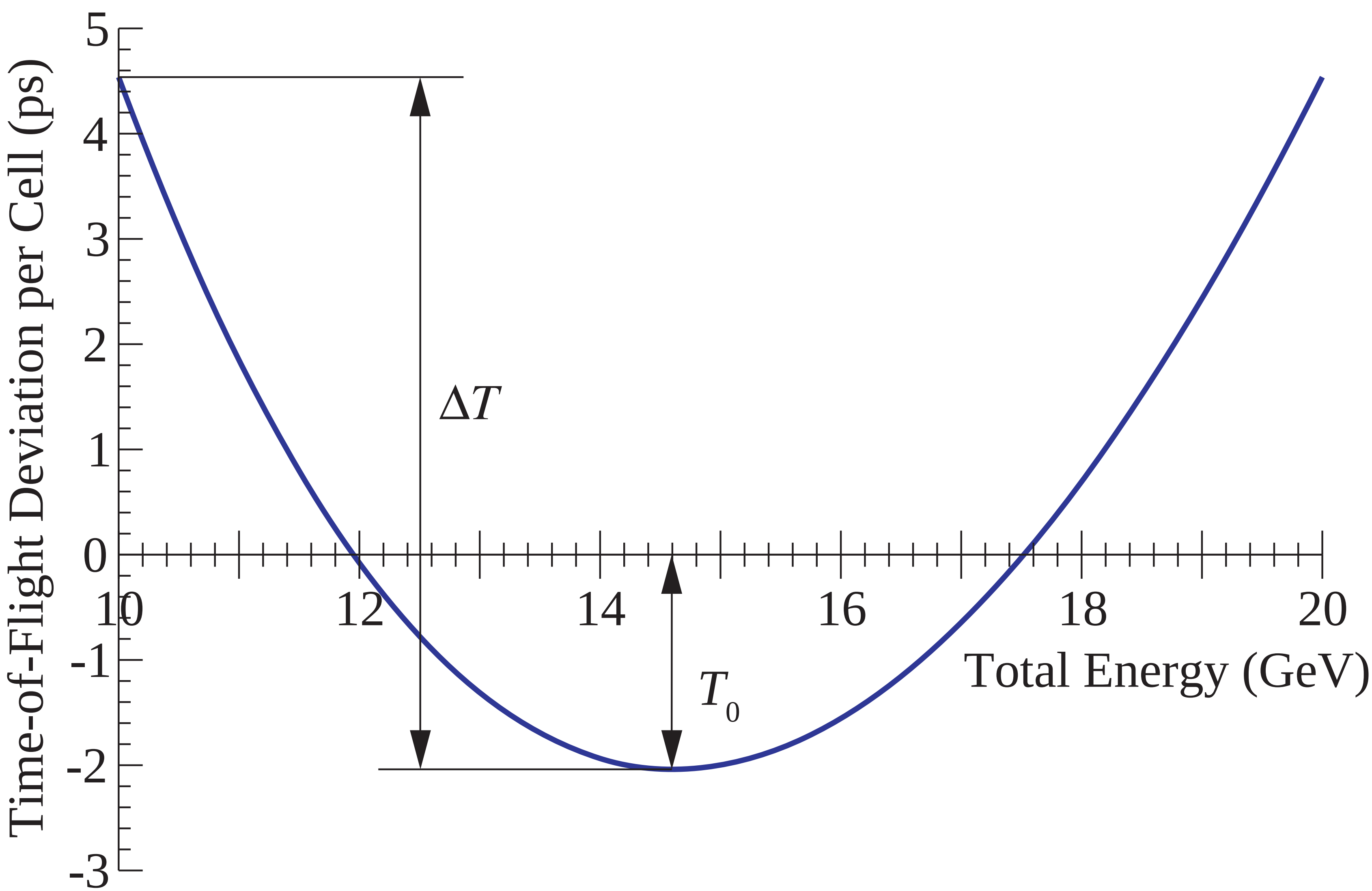
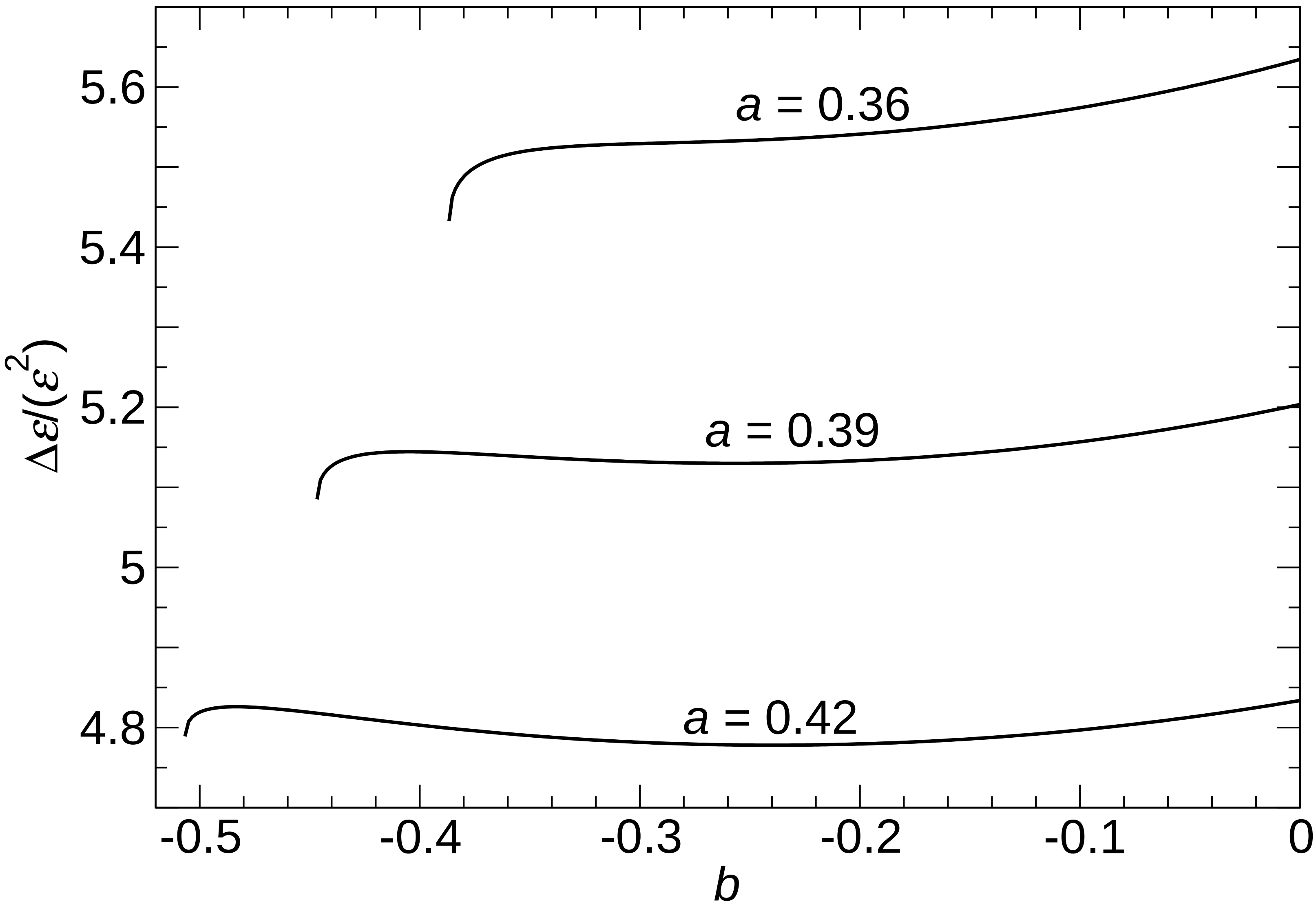


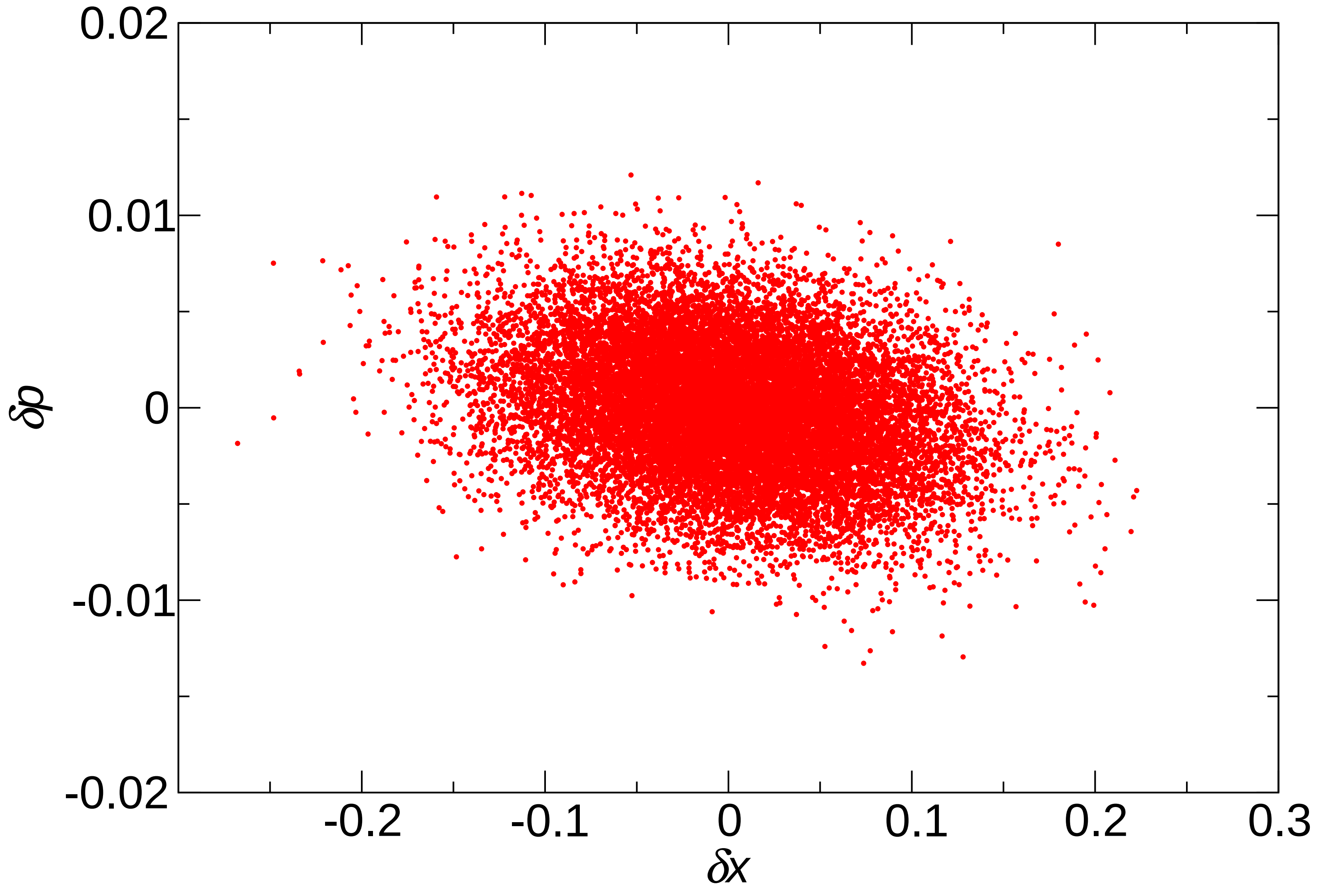
Time-of-Flight vs. Energy



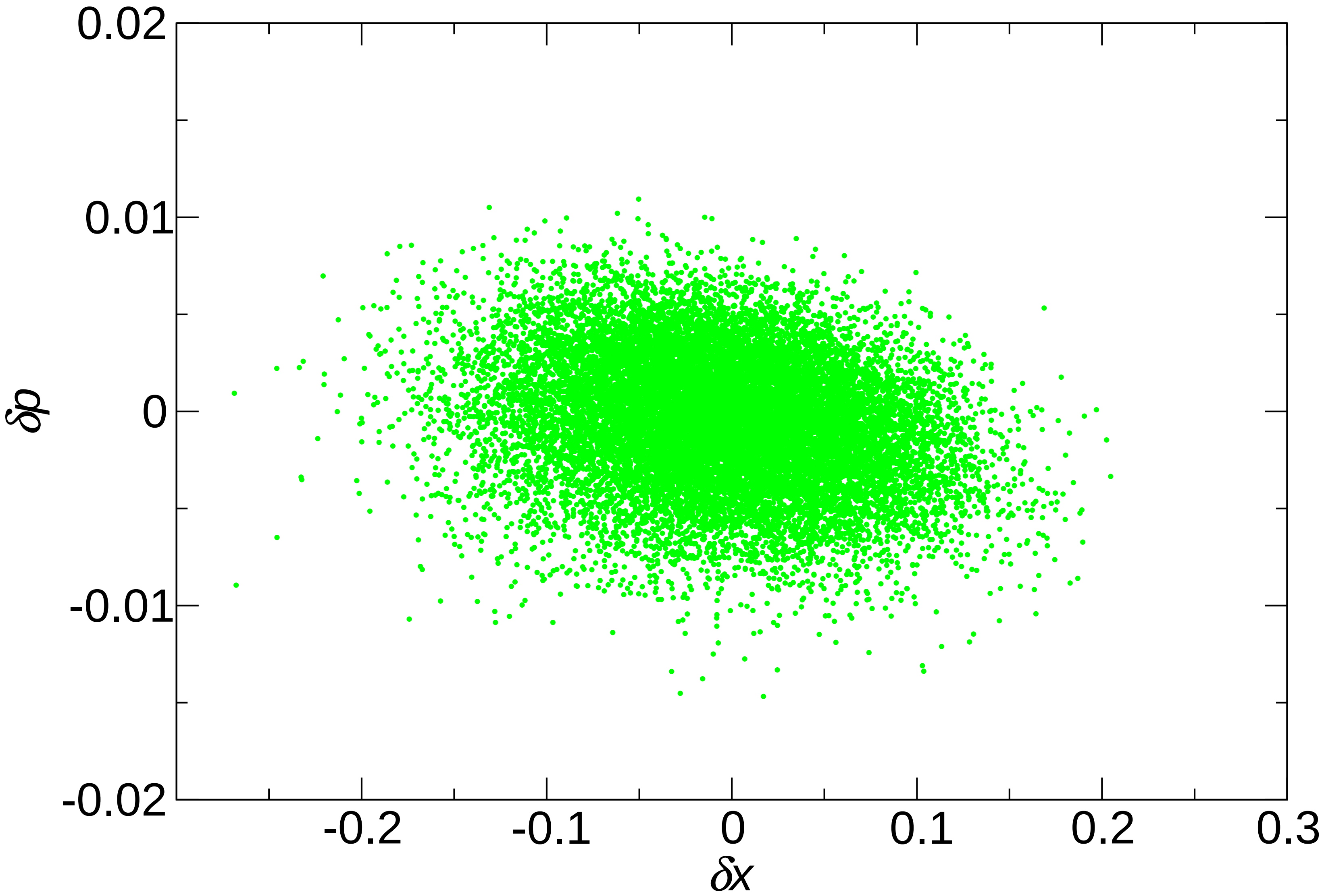
Emittance Growth vs. b



Optimal Orientation



After FFAG



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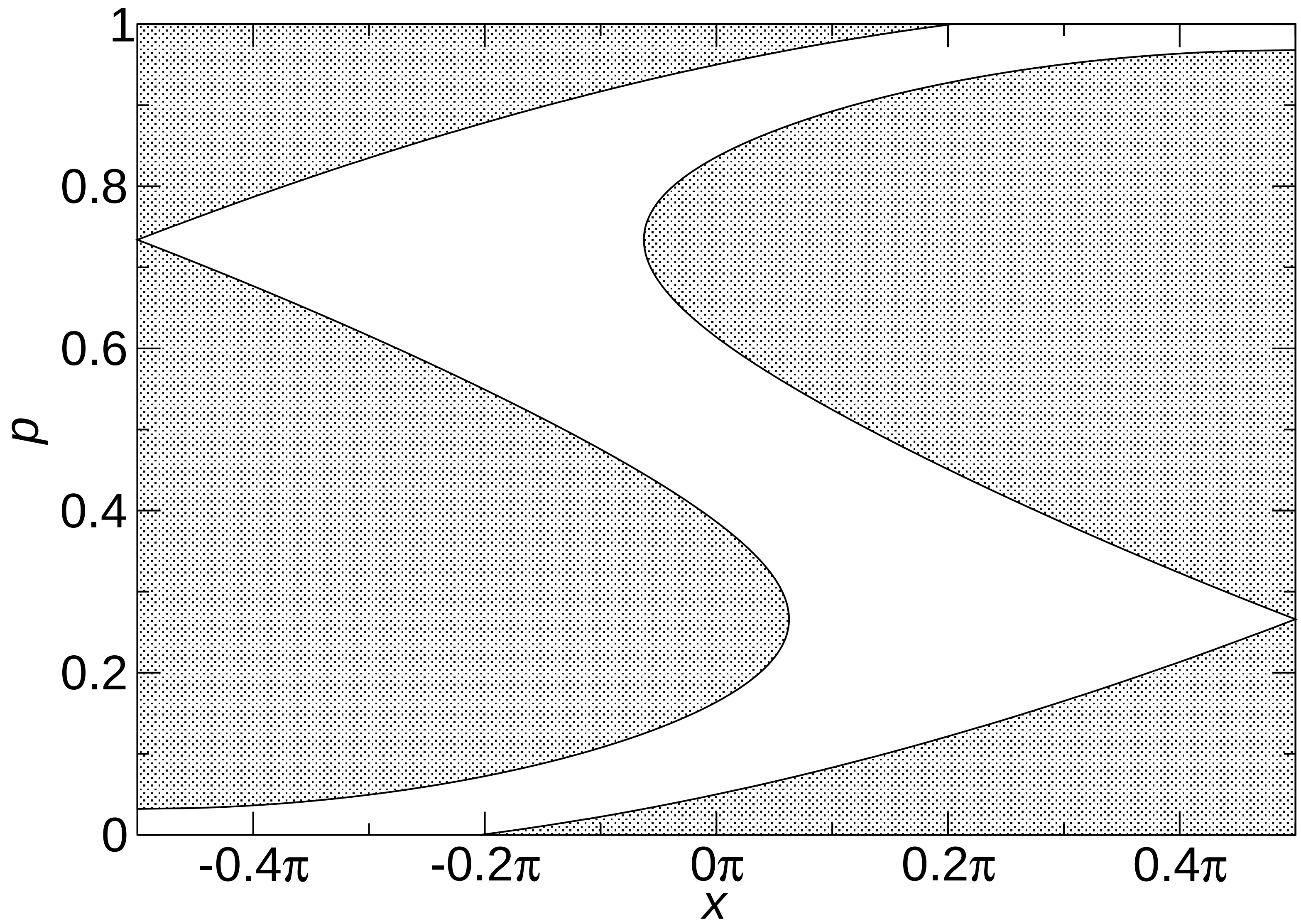

Emittance Reduction Example

- Before, found that for some cases to lowest order, emittance went down!
- What does this mean?
- Properly choose f_3 to get "emittance reduction"
- Nearly uniform distribution, but weighted slightly to the outside. 0.6% emittance reduction
- Distribution more heavily weighted to the outside: 6.3% emittance reduction
- Difficult to get reductions significantly larger than this: would need higher amplitude distributions, and higher order terms start to dominate

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Nearly Uniform: Before

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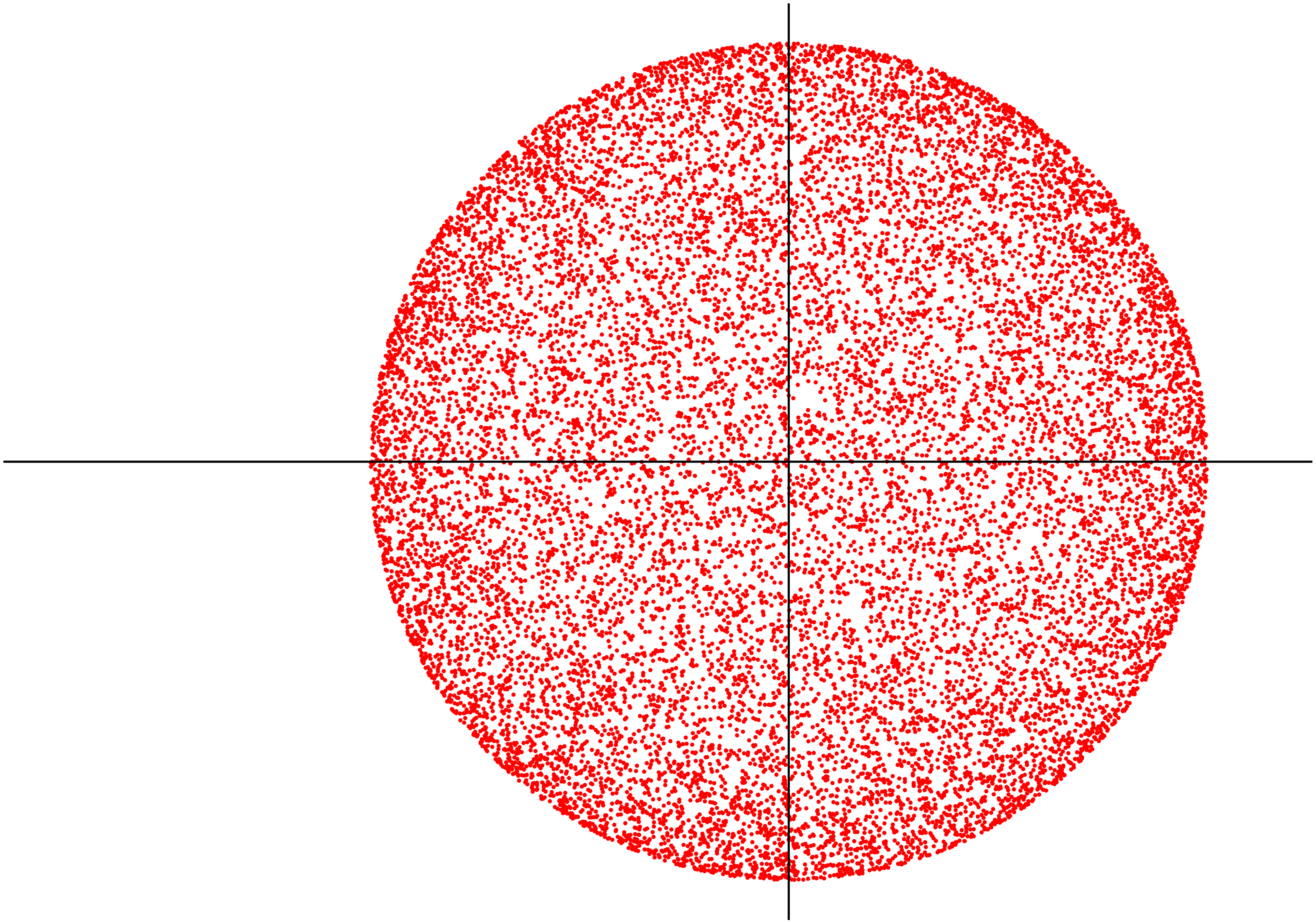
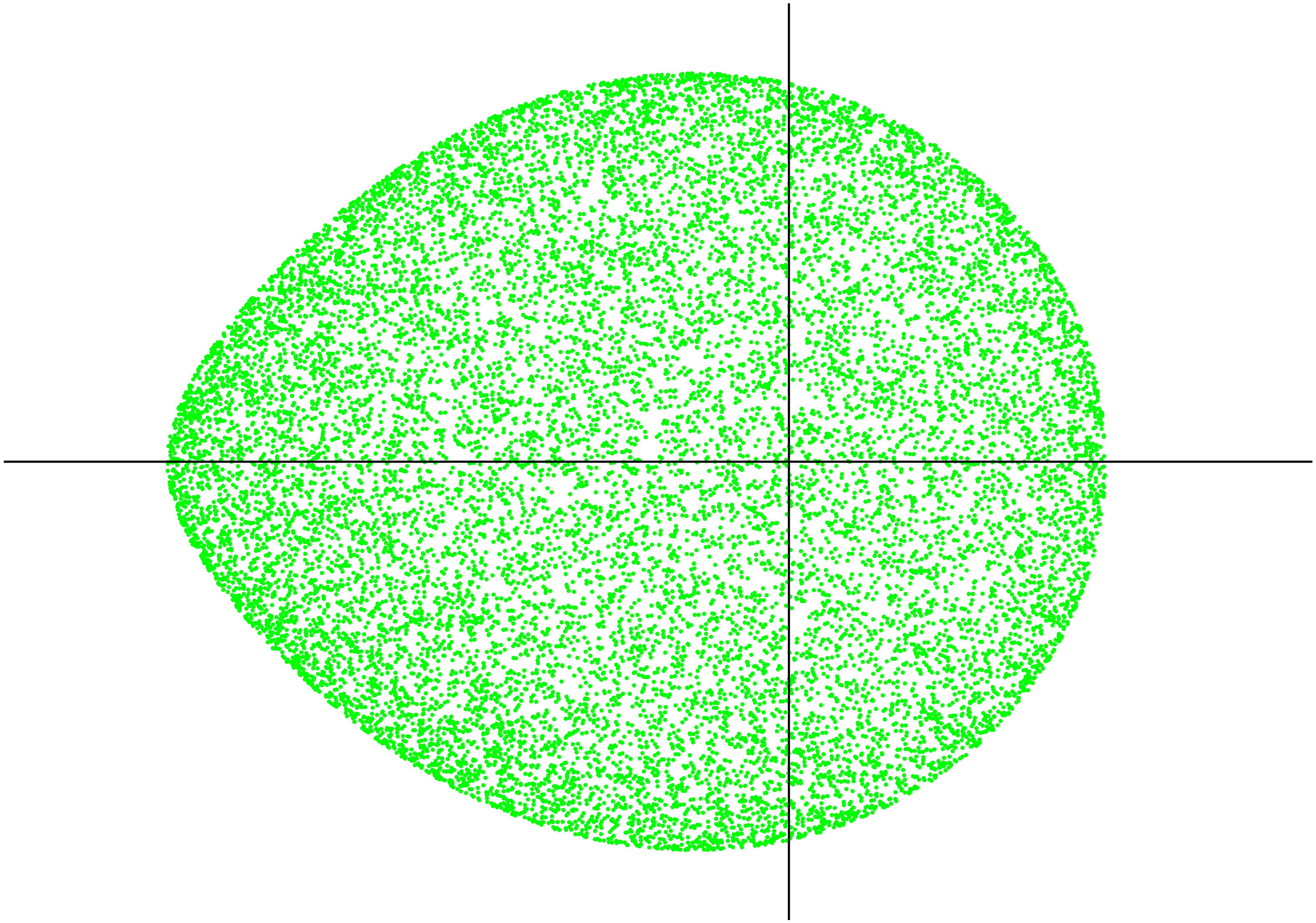
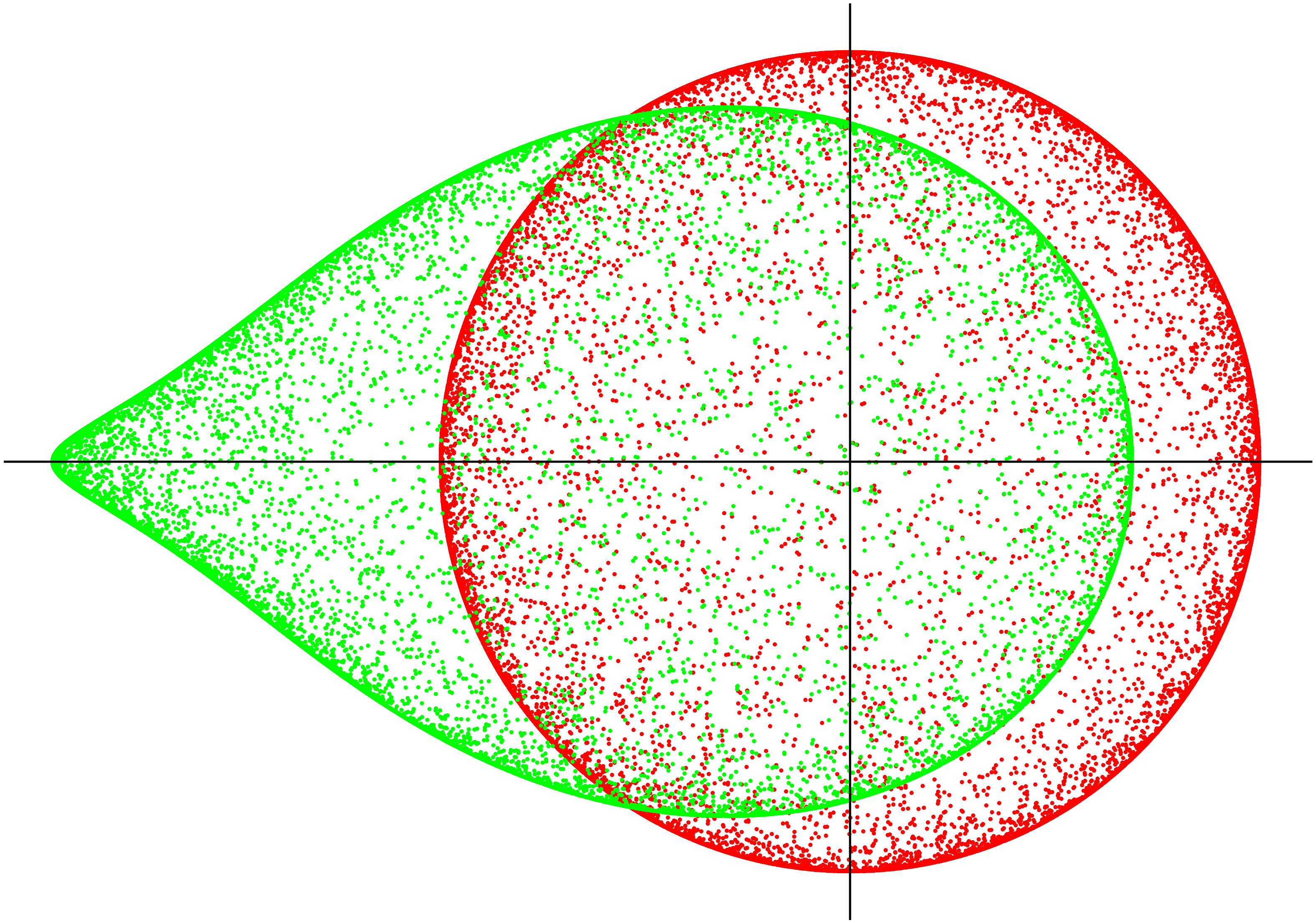


Nearly Uniform: After



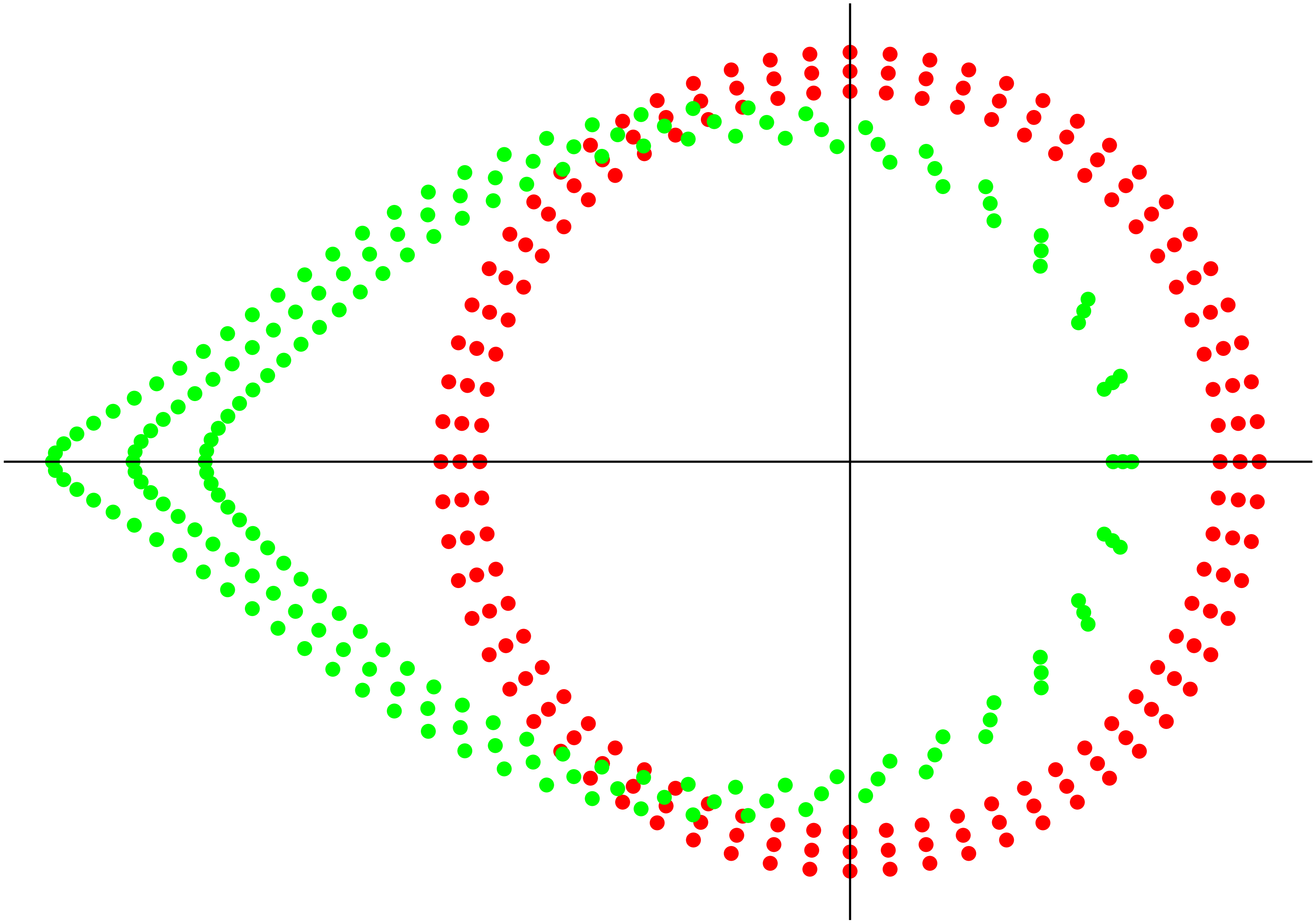
Ring Distribution



Analysis

- Phase space area occupied and local density stay the same! No violation of phase space area conservation
- Distribution is getting nonlinearly shifted toward the left center.
  - Particles are getting concentrated near that point, reducing computed emittance
  - With a more uniform distribution, particles are also pushed away from that point
  - Ring-like distribution has fewer particles being pushed away

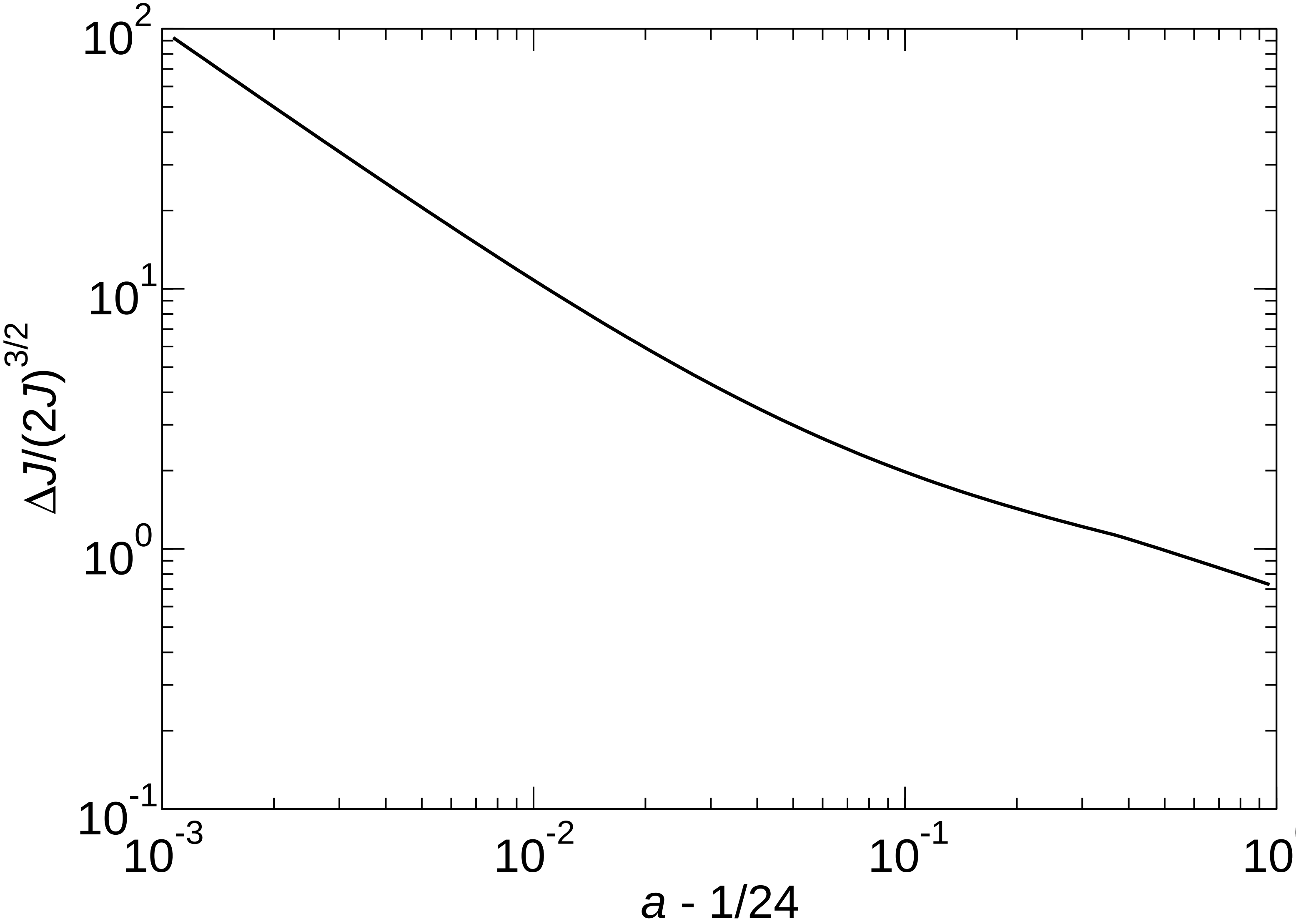
Individual Particles



Ellipse Distortion

- Potentially better criterion for FFAG performance: ellipse distortion
  - Start with an ellipse, measure the deviations from the closest ellipse at end
- As before, plot ellipse distortion vs. a
- Note different qualitative behaviors
  - Emittance growth was proportional to ϵ^2; action distortion is proportional to $(2J)^{3/2}$. Equivalently, radius distortion is proportional to r^2.
  - Coefficient is proportional to $(a - 1/24)^{-1}$, whereas for emittance growth it was $(a - 1/24)^{-2}$

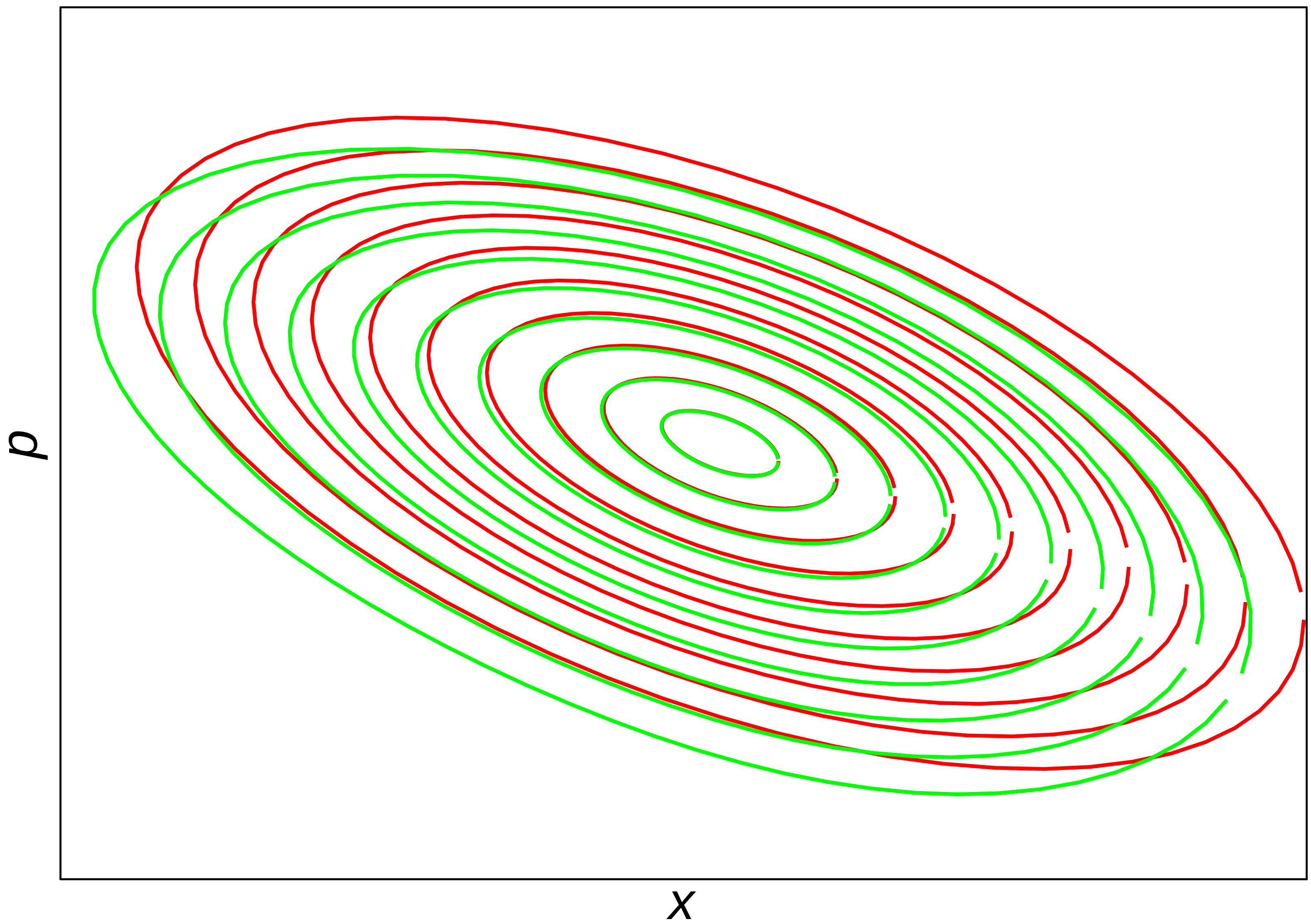
Ellipse Distortion vs. a



Improved Computation

- Leaving out two effects
  - Amplitude-dependent shift of the ellipse center
  - Amplitude-dependent distortion of the ellipse shape
  - If we include these, then we don't care where the center of the ellipse is; we only care about the outer boundary enclosing all particles
- Including these effects, action distortion will be proportional to $(2J)^{5/2}$, or radius distortion proportional to r^4
  - This gives significantly less distortion for small radii
- Good for neutrino factory: don't care what low amplitude particles are doing
- May not be as good for collider
- Still working on the computation. . .

Ellipse Distortion vs. Amplitude



Conclusions

- Have two ways of computing longitudinal phase space distortion for a muon FFAG
  - Emittance growth
  - Ellipse distortion
- Can use these to choose design parameters for an FFAG
- Can include amplitude-dependent shifts in the ellipse distortion computation
  - May give better results for neutrino factory scenario
- For some distributions, nonlinearities alone can lead to reduction of emittance ***as computed using second order covariant matrix***
- This is not a real increase in phase space density: Liouville still holds!